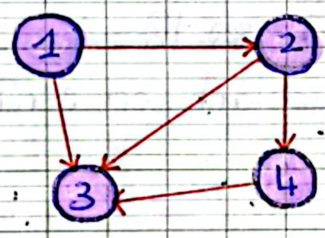
Network Centrality

→ Represent a given table in a graph

Ex:

vertex (node)	vertex	weights (if any)
1	2	30
1	3	5
2	3	22
2	4	2
4	3	37

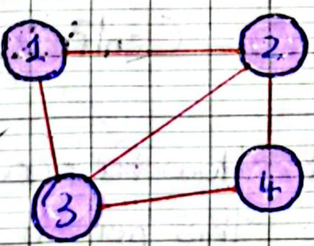
1) Directed Graph



Node	1	2	3	4
1	-	1	1	0
2	0	-	1	1
3	0	0	-	0
4	0	0	1	-

Adjency Matrix

2) Undirected Graph

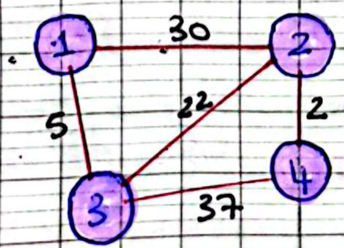


Node	1	2	3	4
1	-	1	1	0
2	1	-	1	1
3	1	1	-	1
4	0	1	1	-

Symetric since undirected

Adjency Matrix

3) Weighted Graph (+ undirected)

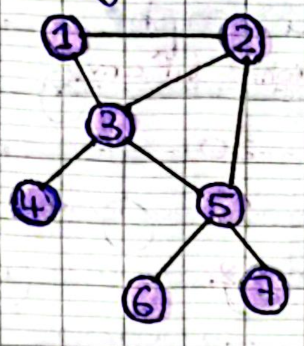


Node	1	2	3	4
1	-	30	5	0
2	30	-	22	2
3	5	22	-	37
4	0	2	37	-

Instead of 1, add the weight

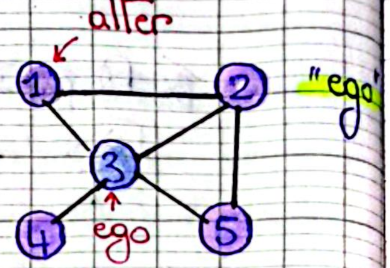
Adjency Matrix

→ "Ego" vs. "whole" Network



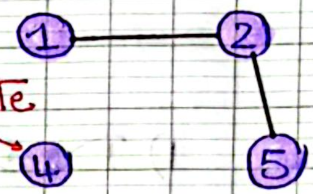
"whole"

If we take ③ as the ego ⇒



⇓

IF we remove ③



⇒ Centrality

- Centrality measures address the question: "who is the most important or central person in this network?"
  - ↳ There are many answers, depending on what we mean by importance

Degree Centrality

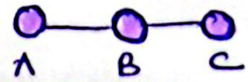
"How many people can this person reach directly?"

↳ (nb of direct connections can a node have)

Betweenness Centrality

"How likely is this person to be the most direct route between 2 people in the network?"

↳ (probability of the node B to be the most direct route between A and C



Closeness Centrality

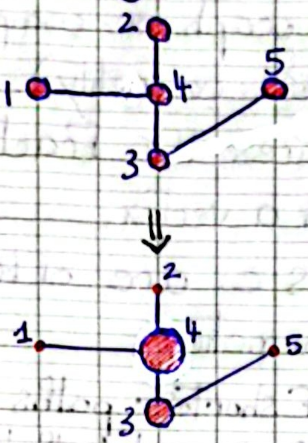
How fast can this person reach everyone in the network?"

Eigenvector

"How well is this person connected to other well-connecte people?"

1) Degree Centrality : $C_{Di} = \frac{\sum a_{ij}}{n-1}$

undirected graph



	1	2	3	4	5	Sum	N-1	Degree
1	-	0	0	1	0	1	4	1/4
2	0	-	0	1	0	1	4	1/4
3	0	0	-	1	1	2	4	2/4 ← Second highest
4	1	1	1	-	0	3	4	3/4 ← highest degree
5	0	0	1	0	-	1	4	1/4

2) Closeness Centrality $C(x) = \frac{N-1}{\sum d(y,x)}$

* Nodes with a high closeness Score have the shortest distances to all other nodes

no repeated verticies in a path

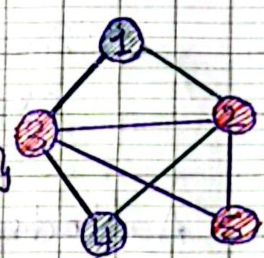
* Path vs Shortest Path

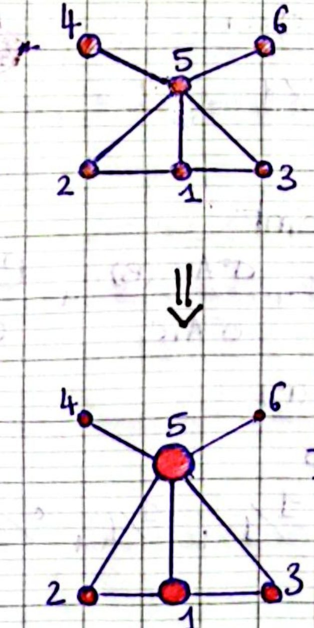
From 1 to 4

→ Shortest Paths: {1,2,4}; {1,3,4}

→ Other Paths:

{1,2,3,4}; {1,3,2,4}; {1,2,5,3,4}





5>1>2>3>4>6

	1	2	3	4	5	6	N-1	Sum	Closeness
1	-	1	1	2	1	2	5	7	5/7 0.7
2	1	-	2	2	1	2	5	8	5/8 0.6
3	1	2	-	2	1	2	5	8	5/8 0.6
4	2	2	2	-	1	2	5	9	5/9 0.5
5	1	1	1	1	-	1	5	5	5/5 1
6	2	2	2	2	1	-	5	9	5/9 0.5

highest closeness

Imp!! here we don't look at the direct connections.

2 ⇒ represent the nb. of connections to reach the Targeted node using the shortest path

From 6 to 1, we path by {6,5} then {5,1}

3) Betweenness Centrality

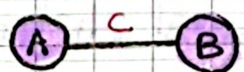
group of nodes connected to each other

- we don't need a fully connected graph or component to calculate it (unlike closeness centrality)
- betweenness centrality measures the fraction of shortest paths passing through a vertex
- Nodes with high betweenness c. are often important controllers of power or information
- $C_{btw}(v) = \sum_{s,t} \frac{\sigma_{s,t}(v)}{\sigma_{s,t}}$

  $\sigma_{s,t}(v)$: nb. of shortest paths between nodes s and t that pass through v

  $\sigma_{s,t}$: nb. of shortest paths between nodes s and t

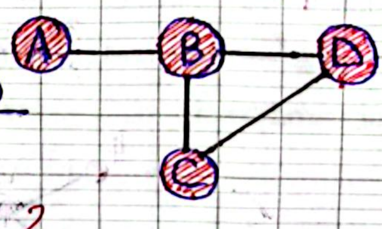
- Endpoint:



A and B are endpoints for the connection c

↳ So in betweenness, we can either include or exclude node v as node s and t in the computation

A_ Undirected Graph



↳ Exclude node v as endpoint

$$C_{btw}(B) = \frac{\sigma_{A,D}(B)}{\sigma_{A,D}} + \frac{\sigma_{A,C}(B)}{\sigma_{A,C}} + \frac{\sigma_{C,D}(B)}{\sigma_{C,D}}$$

$$= 1/1 + 1/1 + 0/1 = 2$$

↳ Include node v as endpoint

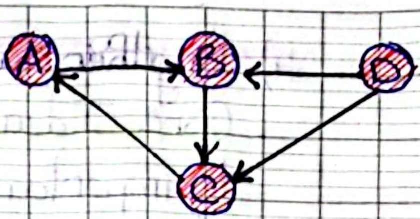
$$C_{btw}(B) = \frac{\sigma_{A,B}(B)}{\sigma_{A,B}} + \frac{\sigma_{A,D}(B)}{\sigma_{A,D}} + \frac{\sigma_{A,C}(B)}{\sigma_{A,C}} + \frac{\sigma_{B,C}(B)}{\sigma_{B,C}} + \frac{\sigma_{B,D}(B)}{\sigma_{B,D}} + \frac{\sigma_{C,D}(B)}{\sigma_{C,D}}$$

$$= 1/1 + 1/1 + 1/1 + 1/1 + 1/1 + 0/1$$

$$= 5$$

B- Directed Graph

↳ Excluding v as endpoint



* $C_{btw}(B) = \frac{\sigma_{A,C}(B)}{\sigma_{A,C}} + \frac{\sigma_{C,A}(B)}{\sigma_{C,A}} + \frac{\sigma_{D,A}(B)}{\sigma_{D,A}} + \frac{\sigma_{D,C}(B)}{\sigma_{D,C}}$

$= 1/1 + 0/1 + 0/1 + 0/1 = 1$

* $C_{btw}(C) = \frac{\sigma_{A,B}(C)}{\sigma_{A,B}} + \frac{\sigma_{B,A}(C)}{\sigma_{B,A}} + \frac{\sigma_{D,B}(C)}{\sigma_{D,B}} + \frac{\sigma_{D,A}(C)}{\sigma_{D,A}}$

$= 0/1 + 1/1 + 0/1 + 1/1 = 2$

* Betweenness Centrality - Normalization

. In graphs with many nodes, betweenness c. values will be large

↳ to control this, we divide centrality values by the nb of pairs of nodes in the graph (excluding v)

. Formula

↳ for undirected graphs : $\frac{1}{2}(|N|-1)(|N|-2)$

↳ for directed graphs: $(|N|-1)(|N|-2)$

. Example:

↳ In undirected graph in the previous example, excluding B

$C_{btw}(B) = 2$

Nb. of paths $= \frac{1}{2}(|N|-1)(|N|-2)$

$= \frac{1}{2}(4-1)(4-2) = 3$

$\Rightarrow$ Normalization Factor $= C_{btw,norm}(B) = \frac{C_{btw}(B)}{(|N|-1)(|N|-2)} = \frac{2}{3}$

↳ In directed graph: $C_{btw}(B) = 1$, Nb. of path $= 3$

$\Rightarrow C_{btw,norm}(B) = 1/3$

4). Eigenvector Centrality

- Core Idea: "You are important if you're connected to important people"
- "Eigen" (German) = "Self" + "vector"
- Def: Centrality score proportional to sum of connected nodes' scores

- Eigenvalue Problem
  - goal: Find λ and x such that $\underbrace{A}_{\text{Adjacency matrix}} \overbrace{x}^{\text{Eigenvector (centrality score)}} = \underbrace{\lambda}_{\text{Eigenvalue}} x$

  - Example:

identity matrix: $n \times n$ matrix with 1s on diagonal and 0s elsewhere

$$\underset{(2\times 2)}{A} = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$$

1) multiply λ by identity matrix (2x2)

$$\lambda I = \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

2) Subtract A from λI

$$A - \lambda I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}$$

3) Calculate the determinant of $A - \lambda I$

$$\det(A - \lambda I) = (7-\lambda)(-1-\lambda) - (3)(3)$$
$$= -7 - 7\lambda + \lambda + \lambda^2 - 9$$
$$= \lambda^2 - 6\lambda - 16$$

4) Solve values of λ that satisfy $\det(A - \lambda I) = 0$

$\lambda^2 - 6\lambda - 16 = 0$

$(\lambda - 8)(\lambda + 2) = 0$

$\lambda - 8 = 0 \qquad \lambda + 2 = 0$

$\boxed{\lambda = 8} \qquad \boxed{\lambda = -2}$

eigenvalues

5) Solve corresponding vector to each λ

. for $\lambda = 8$

$$\begin{bmatrix} 7-8 & 3 \\ 3 & -1-8 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \leftarrow B$$

$\hookrightarrow$ Solve $B\bar{x} = \bar{0}$

$$\begin{array}{l} eq1 \rightarrow \\ eq2 \rightarrow \end{array} \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

eq1 : $-x_1 + 3x_2 = 0$

eq2 : $3x_1 - 9x_2 = 0$

$\hookrightarrow$ both eq are equivalent so, we can solve one of them

$-x_1 + 3x_2 = 0$

$3x_2 - x_1 = 0$

$x_1 = 3 \qquad x_2 = 1$

$\Rightarrow$ so, eigenvector $\bar{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

. for $\lambda = -2$

$$\begin{bmatrix} 7+2 & 3 \\ 3 & -1+2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \leftarrow B$$

. $B\bar{x} = 0$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 9x_1 + 3x_2 = 0$$

$x_1 = 1 \ , \ x_2 = -3$

$\Rightarrow$ for $\lambda = -2$ eigenvector $\bar{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$